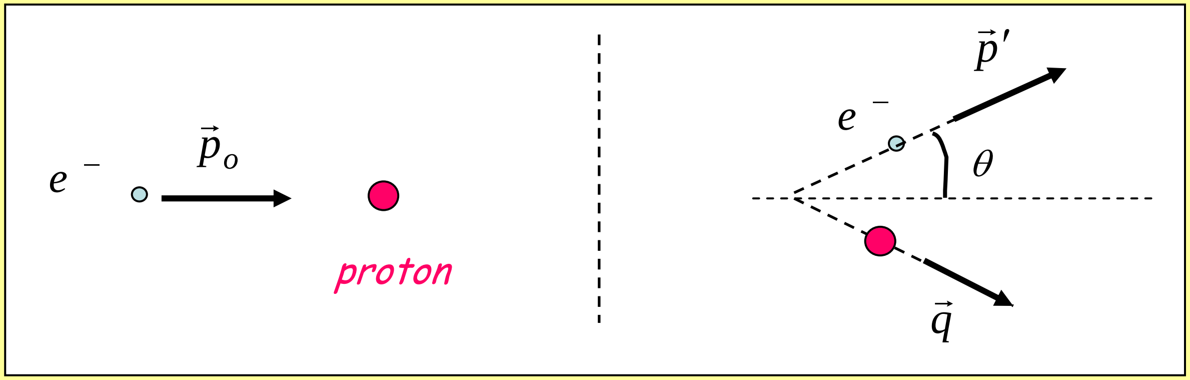
Details, Part III: Kinematics



Electrons are **relativistic** → kinetic energy $K \neq p^2/2m$ ...

Einstein mass-energy relations:

(total energy E, rest mass m)

1) $E^2 = (mc^2)^2 + (cp)^2$

2) $E = mc^2 + K$

Problem: units are awkward, too many factors of c ...

Notice that **if c=1** then (E, m, p, K) all have the same units!

If we set c = 1 in Einstein's mass-energy relations, then in order to "get the answer right", the factor c has to be absorbed in the units of p and m:

$$1)\quad E^2 = m^2 + p^2$$

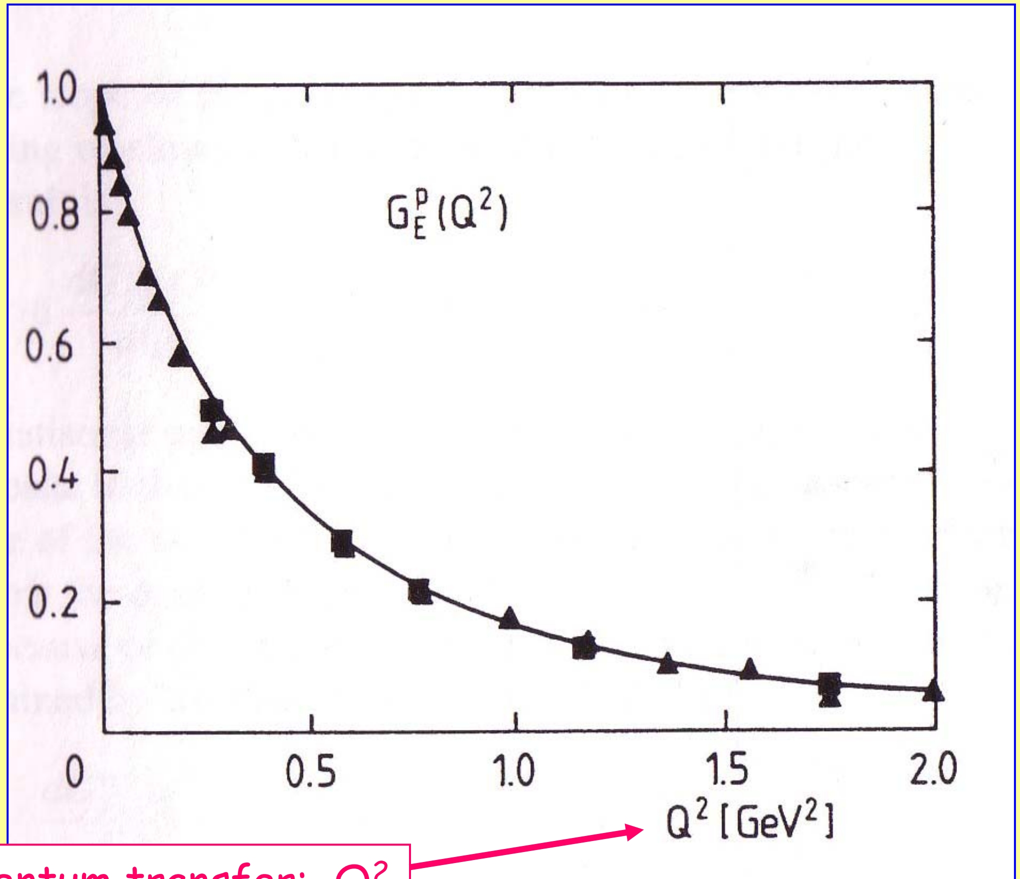
$$2)\quad E = m + K$$

Let the symbol "[ ]" mean "the units of", and then it follows that:

$$[E] = \mathrm{GeV}, \quad [m] = \mathrm{GeV/c}^2, \quad [p] = \mathrm{GeV/c}$$

(Frequently, physicists set c = 1 and quote mass and/or momentum in "GeV" units, as in the graph of the proton electric form factor, lecture 4. This is just a form of shorthand – they really mean GeV/c for momentum and GeV/c^2 for mass.... numerically these have the same value because **the value of c is in the unit** – we don't divide by the numerical value 3.00×10^8 m/s or the answer would be ridiculously small (wrong!)

From lecture 4: proton electric form factor data



4 – momentum transfer: Q^2

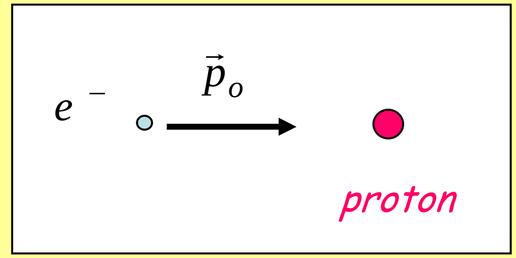
Ref: Arnold et al., Phys. Rev. Lett. 57, 174 (1986)

(Inverse Fourier transform gives the electric charge density $\rho(r)$)

More units....

When we want to describe a scattering problem in quantum mechanics, we have to write down wave functions to describe the initial and final states....

For example, the incoming electron is a free particle of momentum: $\vec{p}_o = \hbar \vec{k}_o$



The electron wave function is: where V is a normalization volume.

$$\psi(\vec{r}) = \frac{1}{\sqrt{V}} e^{i\vec{k}_o \bullet \vec{r}}$$

If we set $\hbar$ = 1, then momentum p and wave number k have the same units, e.g. fm^{-1};

to convert, use the factor:

$$\boxed{\hbar c = 197 \text{ MeV.fm} \rightarrow k = \frac{cp}{\hbar c}}$$

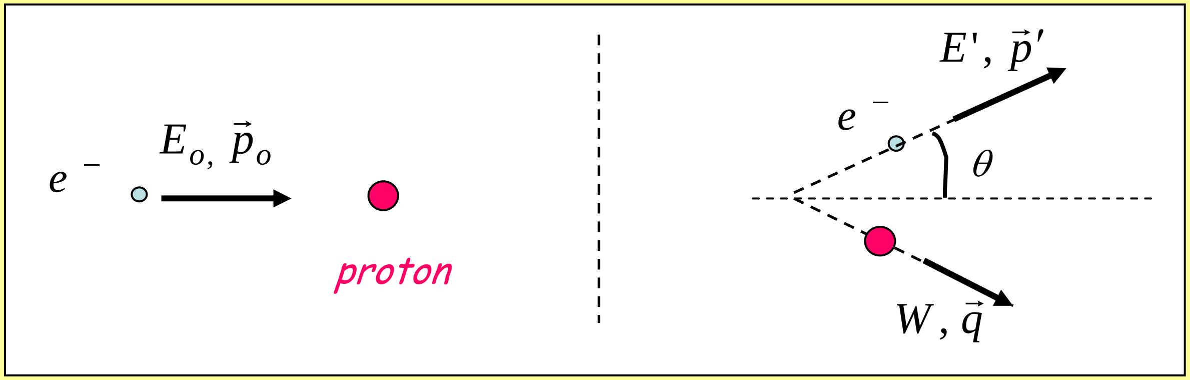
Example:

An electron beam with total energy **E = 5 GeV** has momentum **p = 5 GeV/c** (m << E) ...
the same momentum is equivalent to 5 GeV/(0.197 GeV.fm) or **p = k = 25 fm^{-1}**.

So, p = 5 GeV, 5 GeV/c, and 25 fm^{-1} all refer to the same momentum!

Analysis: Kinematics of electron scattering

Note: Elastic scattering is the relevant case for our purposes here. This means that the beam interacts with the target proton with no internal energy transfer.



- Specify total energy and momentum for the incoming and outgoing particles as shown.
- Electron mass m << E_o. Proton mass is M.

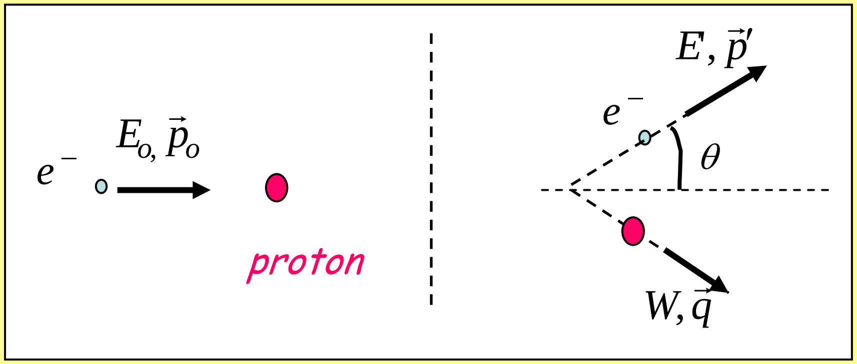
Conserve total energy and momentum: $\boxed{E_o + M = E' + W, \quad \vec{p}_o = \vec{p}' + \vec{q}}$

Next steps: find the scattered electron momentum p′ in terms of the incident momentum and the scattering angle. Also, find the momentum transfer q^2 as a function of scattering angle, because q^2 will turn out to be an important variable that our analysis of the scattering depends on ...

- conserve momentum:

$$\boxed{\vec{q} = (\vec{p}_o - \vec{p}\,')}$$

$$\boxed{1)\ \ q^2 = p_o^2 + (p')^2 - 2p_o\, p' \cos\theta}$$



- now use conservation of energy with $W = M + K$ for the proton; $E = p$ for electrons:

$$\boxed{E_o + M = E' + M + K \quad \Rightarrow \quad K = (p_o - p')}$$

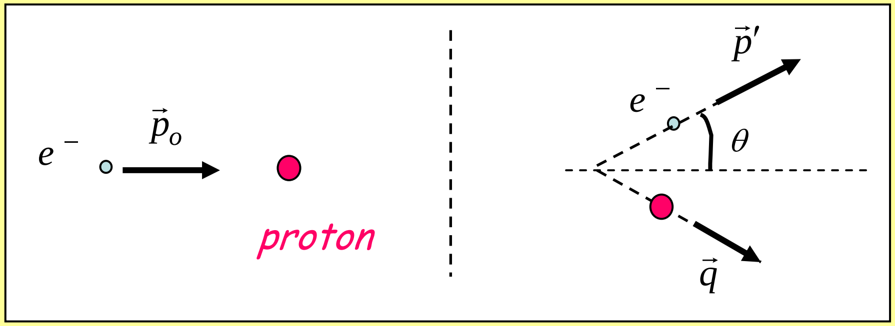
- use kinematic relations to substitute for $\boldsymbol{K = (p_o - p')}$ and q^2:

$$\boxed{W^2 \equiv \cancel{M}^2 + q^2 = \cancel{M}^2 + 2MK + K^2}$$

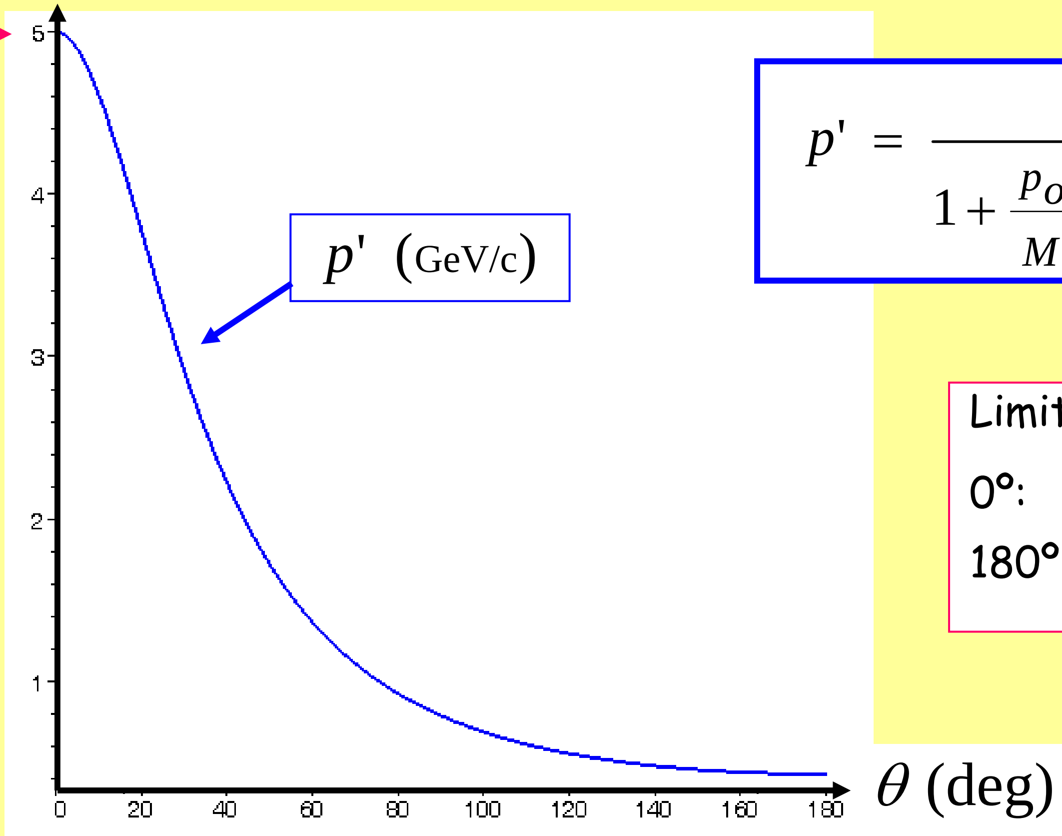
Note: these solutions 1), 2) are perfectly general as long as the electron is relativistic. The target can be anything!

$$\boxed{2)\ \ p' = \frac{p_o}{1 + \frac{p_o}{M}(1 - \cos\theta)}}$$

Example: 5 GeV electron beam, proton target



$p_o = 5\ \mathrm{GeV/c}$



$$p' = \frac{p_o}{1 + \frac{p_o}{M}(1 - \cos\theta)}$$

Limits:

0°: $p' = p_o$

180°: $p' \cong p_o/(1+2p_o)$

It is often convenient to use 4 – vector quantities to work out reaction kinematics.

There are several conventions in this business, all of them giving the same answer but via a slightly different calculation. We will follow the treatment outlined in Perkins, `Introduction to High Energy Physics', Addison-Wesley (3rd Ed., 1987).

Define the relativistic 4-momentum:

$$P_\mu = (\vec{p},\ iE), \qquad \mu = 1 \ldots 4$$

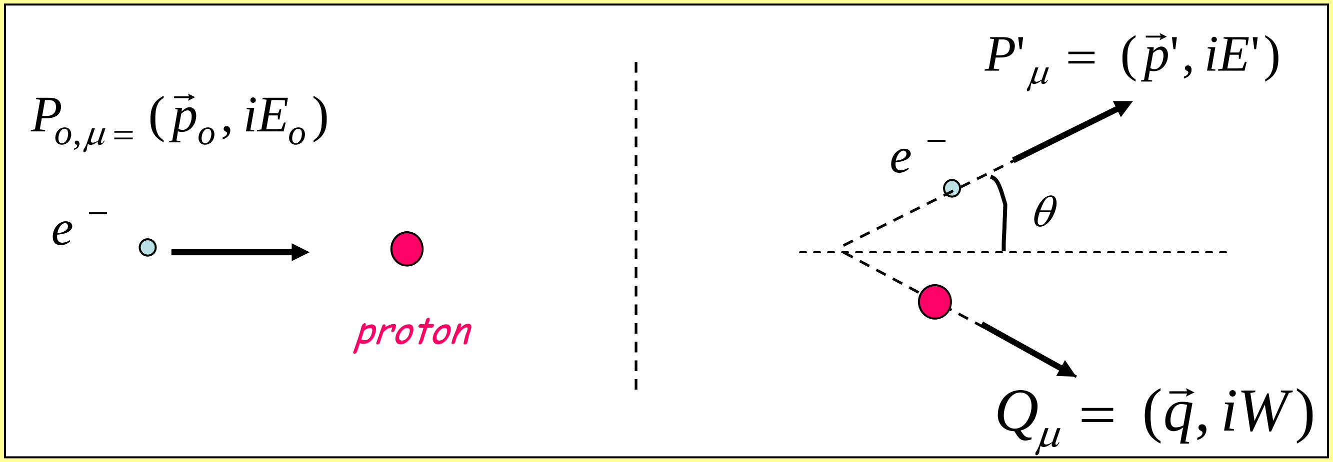
The length of any 4 – vector is the same in all reference frames:

$$\text{`length' squared} \longrightarrow \sum_\mu P_\mu^2 = p^2 - E^2 = -m^2$$

For completeness, a Lorentz boost corresponding to a relative velocity β along the x-axis is accomplished by the 4x4 "rotation" matrix Λ, with:

$$\beta = \mathrm{v}/c, \quad \gamma = (1-\beta^2)^{-1/2}$$

$$\Lambda = \begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$$



<u>Define</u>: 4 - momentum transfer Q between incoming and outgoing **electrons:**

$$Q = (P_o - P') = \left(\vec{p}_o - \vec{p}',\ i\,(E_o - E')\right) = \left(\vec{q}, i\,(E_o - E')\right)$$

Since Q is a 4 – vector, the square of its length is invariant:

$$Q^2 = (\vec{p}_o - \vec{p}')^2 - (E_o - E')^2$$

Expand, simplify, remembering to use $p^2 - E^2 = - m^2$ and $m \ll E,p$ ...

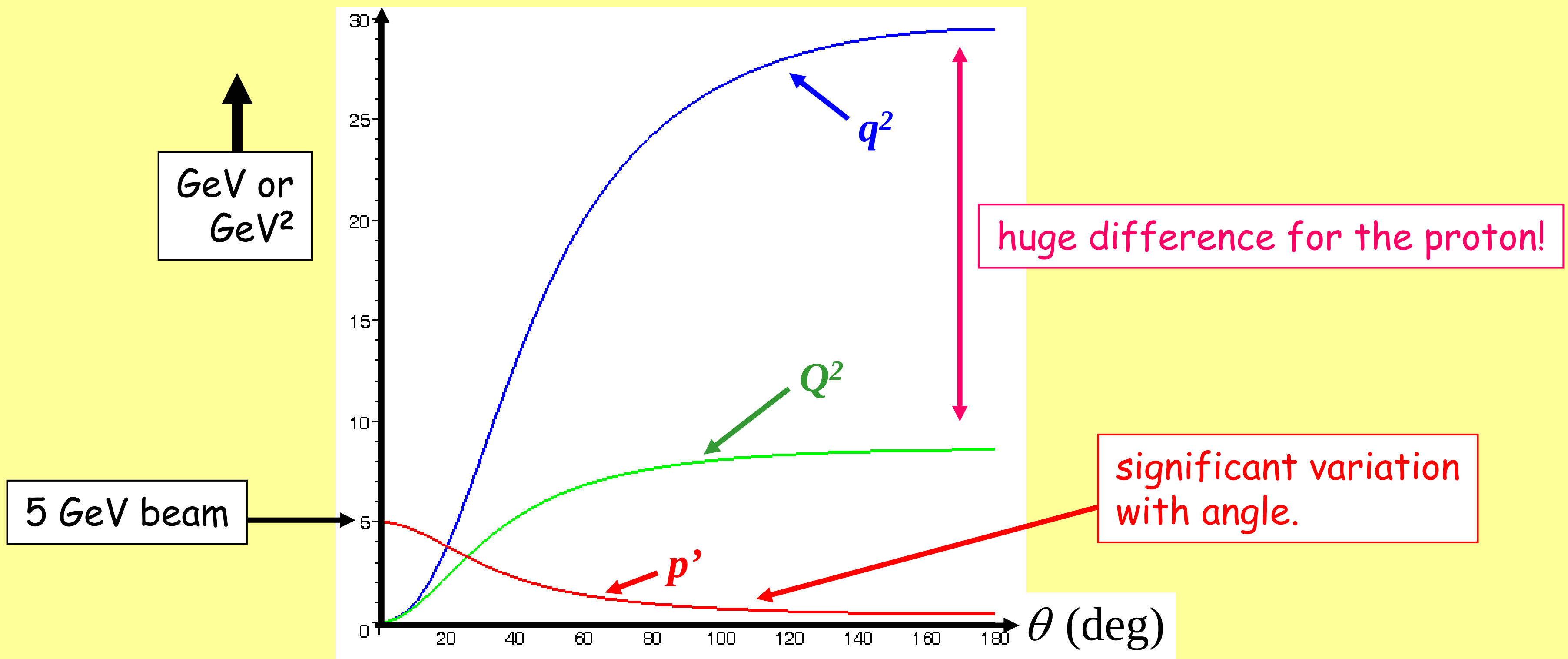
$$\boxed{Q^2 = 2 p_o p' (1 - \cos\theta)}$$

4-momentum transfer squared is the variable used to plot high energy electron scattering data:

$$Q^2 = 2 p_o p' (1 - \cos\theta)$$

For a nonrelativistic quantum treatment of the scattering process (next topic!) the form factor is expressed in terms of the **3-momentum transfer** (squared):

$$q^2 = p_o^2 + (p')^2 - 2 p_o \, p' \cos\theta$$



Note on 4 – vector conventions:

Fraunfelder & Henley use

$$P_\mu = (E, \vec{p}) \quad \rightarrow \quad P^2 = E^2 - p^2 = m^2$$

… so with this convention, one has to define a dot product differently, essentially putting in the minus sign by hand.

Our convention is less arbitrary:

$$\boxed{P_\mu = (\vec{p},\ iE) \rightarrow P^2 = p^2 - E^2 = -m^2}$$

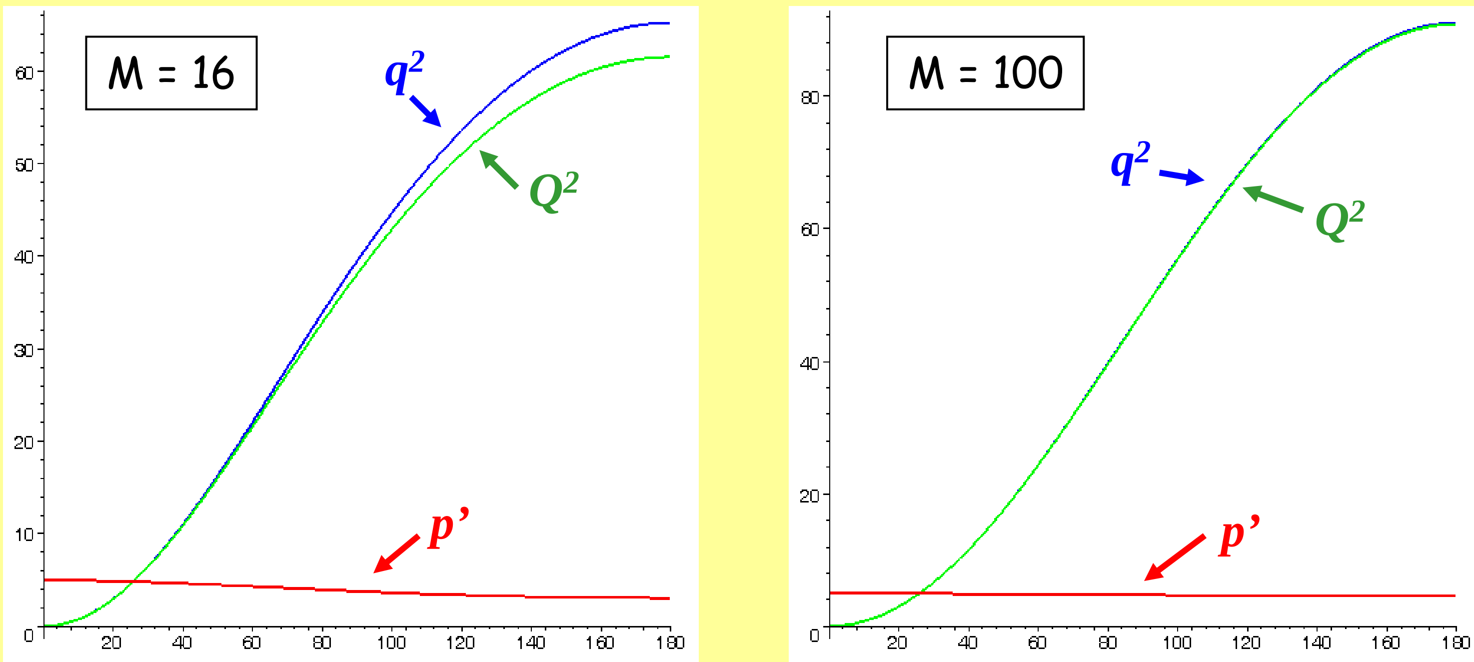
Note that the norms differ by a minus sign, which carries over to the sign of Q^2

(see the note on p. 139, F&H:

their $Q^2 = -p^2$ at low energy, whereas for us, $Q^2 = +p^2$ in the same regime.

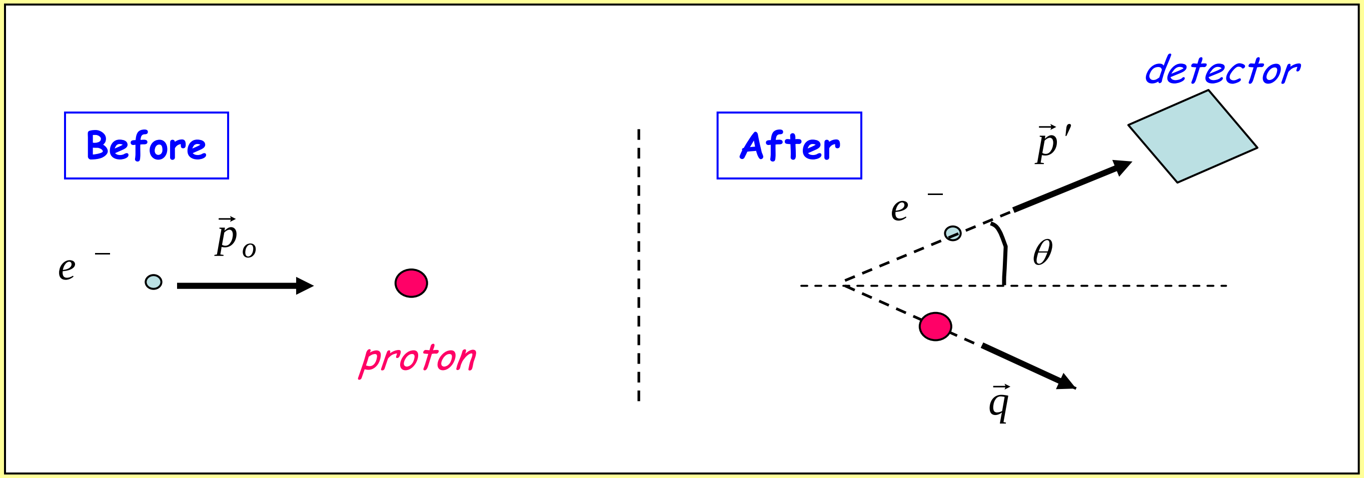
It seems more natural for Q^2 to be a positive number, so our convention is preferred.)

The difference between numerical values of Q^2 and q^2 decreases as the mass of the target increases $\rightarrow$ we can "get away" with a 3-momentum description (easier) to derive the cross section for scattering from a nucleus. Note also the simplification that $p' \cong p_o$ and becomes essentially independent of θ as the mass of the target increases. *(Why? as $M \rightarrow \infty$, the electron beam just 'reflects' off the target – just like an elastic collision of a ping pong ball with the floor – 16.105!!)*



5 GeV electron beam in both cases, as before, but the target mass increased from a proton to a nucleus (e.g., ^{16}O, ^{100}Ru)

Recall from lecture 4:



Experimenters detect elastically scattered electrons and measure the cross section:

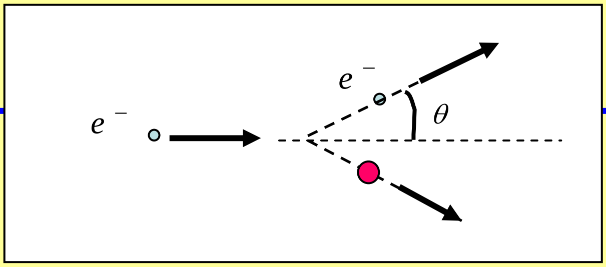
$$\frac{d\sigma}{d\Omega}(\theta) = \left.\frac{d\sigma}{d\Omega}\right)_o \left[F(q^2)\right]^2$$

point charge result (known)

"Form factor" gives the Fourier transform of the extended target charge distribution. Strictly correct for heavy nuclei: same idea but slightly more complicated expression for the proton...

Last job: we want to work out an expression for the scattering cross section to see how it relates to the structure of the target object.

Scattering formalism: *(nonrel. Q.M.)*



Basic idea:

The scattering process involves a transition between an initial quantum state: $|i$ = incoming e^-, target p$\rangle$ and a final state $|f$ = scattered e^-, recoil p $\rangle$.

The transition rate λ_{if} can be calculated from "Fermi's Golden Rule", a basic prescription in quantum mechanics: (ch. 2)

$$\boxed{\lambda_{if} = \frac{2\pi}{\hbar} \left|M_{if}\right|^2 \rho_f}$$

Units: s^{-1}

where the `**matrix element M$_{if}$**' is given by:

$$\boxed{M_{if} = \int \psi_f^* \, V(\vec{r}) \, \psi_i \, d^3 r}$$

The **potential V(r)** represents the interaction responsible for the transition, in this case electromagnetism (Coulomb's law!),

and the `**density of states**' ρ_f is a measure of the number of equivalent final states per unit energy interval – the more states available at the same energy, the faster the transition occurs.

$$\boxed{\rho_f = dn / dE_f}$$

Job for next week: relate the transition rate to the scattering cross section!